Addendum: "On non-equivalence of Lorenz and Coulomb gauges within classical electrodynamics"

[AFLB, Vol. 27, 163, 2002]

Vladimir Onoochin

"Sirius", Nikoloyamski lane 3A, Moscow, 109004, Russia.
E-mail: onoochin@bk.ru

In the mentioned paper [1] it was questioned well known theorem of the classical electrodynamics stating that the expressions for the EM fields evaluated in *any* gauge are *identical*. Because the results of [1] contradict well-established opinion on this point, a short time later Hnizdo [2] showed that at least in direction of motion of the charge, the electric fields in both gauges should be equal. However, despite this work contains the closed-form calculations of identity of the longitudinal component of $\boldsymbol{E}$ fields created by uniformly moving charge, these calculations are made disregarding contraction of the charge while it moves [1].

It should be noted that the paper [1] contains some mistakes. For example, Eqs. (2.16) of [1] is performed without considering the retardation properties of the integrand, and the correct result must be equal to that one given in [4] (Eqs. (15-7.3) to (15-7.5)). But presence of this mistake does not influence the basic statement of the mentioned paper, *i.e.* the expressions for the longitudinal component of Coulomb– and Lorenz–gauges electric fields are *not identical.* However, the essential disadvantage of [1] is that this basic statement is not confirmed by explicit calculations. Here, we cover this gap. Moreover, we show non-identity of expression for any components of the Coulomb– and Lorenz–gauges electric fields.

The simplest way to verify if the gauges are equivalent is to find a connection between the potentials of the Coulomb and Lorenz gauges. Assuming the electric fields in these gauges are equal, we re-write a

[1] In further work on this subject [3], Hnizdo considers the charge of contracted shape.

difference in these fields via the potentials

$$\boldsymbol{E}_{\mathrm{L}} - \boldsymbol{E}_{\mathrm{C}} = 0 \quad \Longrightarrow$$
$$-\boldsymbol{\nabla}\Phi_{\mathrm{L}} + \boldsymbol{\nabla}\Phi_{\mathrm{C}} - \frac{1}{c}\frac{\partial \boldsymbol{A}_{\mathrm{L}}}{\partial t} + \frac{1}{c}\frac{\partial \boldsymbol{A}_{\mathrm{C}}}{\partial t} = 0 \quad . \tag{1}$$

Using Eq. (3.3) of [1] for difference in the Coulomb– and Lorenz–gauge vector potentials, the above equation reads

$$\boldsymbol{\nabla}\Phi_{\mathrm{C}} - \boldsymbol{\nabla}\Phi_{\mathrm{L}} - \frac{\partial}{\partial t}\left[\frac{1}{4\pi c^2}\int G(\boldsymbol{r}-\boldsymbol{r}';t-t')\boldsymbol{\nabla}_{r'}\frac{\partial \Phi_{\mathrm{V}}(\boldsymbol{r}',t')}{\partial t'}\mathrm{d}\boldsymbol{r}'\mathrm{d}t'\right] = 0 \quad \Longrightarrow$$
$$\boldsymbol{\nabla}\Phi_{\mathrm{C}} - \boldsymbol{\nabla}\Phi_{\mathrm{L}} - \boldsymbol{\nabla}\left[\frac{1}{4\pi}\int G(\boldsymbol{r}-\boldsymbol{r}';t-t')\frac{\partial^2 \Phi_{\mathrm{C}}(\boldsymbol{r}',t')}{c^2\partial t'^2}\mathrm{d}\boldsymbol{r}'\mathrm{d}t'\right] = 0 \ . \tag{2}$$

where $G(\boldsymbol{r}-\boldsymbol{r}';t-t')$ is the Green function of the wave equation. One can see that by introducing the quantity

$$\Phi_{\varphi} = -\frac{1}{4\pi}\int G(\boldsymbol{r}-\boldsymbol{r}';t-t')\frac{\partial^2 \Phi_{\mathrm{V}}(\boldsymbol{r}',t')}{c^2\partial t'^2}\mathrm{d}\boldsymbol{r}'\mathrm{d}t' \ ,$$

Eq. (2) can be written as a sum of gradients of the potentials Φ_{C}, Φ_{L} and Φ_{φ}, and we are able to use, instead of Eq. (2), the equation for the potentials only.

$$\Phi_{\mathrm{C}} - \Phi_{\mathrm{L}} + \Phi_{\varphi} = 0 \ . \tag{3}$$

Correctness of the above equation is a criterion of the equivalence of the gauges.

A value of Φ_{φ} cannot be evaluated in the general case, *i.e.* if this potential is created by arbitrary moving elementary charge. But if the charge moves uniformly, for example, along the x-axis, this value can be calculated in the closed form and, therefore, Eq. (3) can be verified. Below we realize this procedure.

Using the "present time" expressions (Ch. 18.3 of [5]) for the potentials entering into Eq. (3), we have

$$0 = \frac{q}{\sqrt{\kappa^2+y^2+z^2}} - \frac{q}{\sqrt{\kappa^2+(1-v^2/c^2)[y^2+z^2]}} - \tag{4}$$
$$-\frac{v^2}{4\pi c^2}\int \frac{\partial^2\Phi_{\mathrm{C}}(\kappa',y',z')}{\partial \kappa'^2}\frac{\mathrm{d}\kappa'\mathrm{d}y'\mathrm{d}z'}{\sqrt{(\kappa-\kappa')^2+(1-v^2/c^2)[(y-y')^2+(z-z')^2]}} \ .$$

where $\kappa = x - vt$. We need to calculate the second partial spatial derivative of Φ_{C} created by the elementary charge whose shape is ellipsoid contracted in direction of motion. This derivative is a regular function ($\frac{\partial^2 \Phi_{\rm C}}{\partial \kappa^2} = -q\frac{r^2-3\kappa^2}{r^5}$) in the exterior region of the charge, however, according to Eq. (18) of Ch. IV.5.5 of [6], improper integral

$$\frac{\partial^2 \Phi_{\rm C}}{\partial x^2} = \frac{\partial^2}{\partial x^2}\int \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}{\rm d}\boldsymbol{r}' =$$
$$\int \rho(\boldsymbol{r}')\frac{\partial^2}{\partial x^2}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}{\rm d}\boldsymbol{r}' = -\int \rho(\boldsymbol{r}')\left(\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|^3} - 3\frac{(x-x')^2}{|\boldsymbol{r}-\boldsymbol{r}'|^5}\right){\rm d}\boldsymbol{r}'. \quad (5)$$

does not converge absolutely for internal points of the charge.

To calculate this improper integral and then the integral in Eq. (4), we separate the region of integration onto the region interior the charge, V_{ch}, where value of ρ is non-zero, and the region exterior the charge, $V - V_{ch}$, where $\rho = 0$. So Eq. (4) is written as

$$0 = \frac{q}{\sqrt{\kappa^2+y^2+z^2}} - \frac{q}{\sqrt{\kappa^2+(1-v^2/c^2)[y^2+z^2]}} -$$
$$-\frac{qv^2}{4\pi c^2}\lim_{V_{ch}\to 0}\int\limits_{V_{ch}}^{\infty} \frac{3{\kappa'}^2 - {r'}^2}{\left[\sqrt{{\kappa'}^2+{y'}^2+{z'}^2}\right]^5}\frac{{\rm d}\kappa'{\rm d}y'{\rm d}z'}{\sqrt{(\kappa-\kappa')^2+b^2[(y-y')^2+(z-z')^2]}} +$$
$$+\frac{qv^2}{4\pi c^2}\lim_{V_{ch}\to 0}\int\limits_{0}^{V_{ch}}\left[\frac{\partial^2\Phi_{\rm C}}{\partial {\kappa'}^2}\right]_{in}\frac{{\rm d}\kappa'{\rm d}y'{\rm d}z'}{\sqrt{(\kappa-\kappa')^2+b^2\left((y-y')^2+(z-z')^2\right)}}\ , \quad (6)$$

where $b = \sqrt{1-v^2/c^2}$ and $\left[\frac{\partial^2\Phi_{\rm C}}{\partial\kappa^2}\right]_{in}$ is some yet unknown function, which is defined in V_{ch}. Now we need to evaluate the integrals $I_{\varphi}(\boldsymbol{r})$ and $I_{\partial}(\boldsymbol{r})$ in the second and third lines of (6) ($\Phi_{\varphi}(\boldsymbol{r}) = I_{\varphi}(\boldsymbol{r}) + I_{\partial}(\boldsymbol{r})$) but because a procedure of evaluation is sufficiently complicated all its details are entered in *Appendix A* and *Appendix B.* As one can see from the final expressions for these integrals, both $I_{\varphi}(\boldsymbol{r})$ and $I_{\partial}(\boldsymbol{r})$ are the functions of the contracted coordinates

$$\mathrm{x} = \kappa/\sqrt{1-v^2/c^2}\ ,\ \mathrm{y} = y\ ,\ \mathrm{z} = z \quad \text{and} \quad \mathrm{r} = \sqrt{\mathrm{x}^2+\mathrm{y}^2+\mathrm{z}^2}\ . \quad (7)$$

So re-writing $\Phi_{\rm C}$ and $\Phi_{\rm L}$ via these coordinates too,

$$\Phi_{\rm C} = \frac{q}{\sqrt{\kappa^2+y^2+z^2}} = \frac{q}{\sqrt{1-(v^2/c^2)\xi^2}\mathrm{r}}$$
$$\Phi_{\rm L} = \frac{q}{\sqrt{\kappa^2+(1-v^2/c^2)[y^2+z^2]}} = \frac{q}{\sqrt{1-(v^2/c^2)}\mathrm{r}}$$

where $\xi = \mathrm{x}/\mathrm{r}$, and using the final results (31) and (45) of *Appendix*es, we have for Eq. (6)

$$\Phi_{\mathrm{L}}(\boldsymbol{r}) - \Phi_{\mathrm{C}}(\boldsymbol{r}) - I_{\partial}(\boldsymbol{r}) - I_{\varphi}(\boldsymbol{r}) = 0 \qquad \Longrightarrow$$
$$\left[\frac{1}{\sqrt{1-v^2/c^2}} - \frac{1}{\sqrt{1-(v^2/c^2)\xi^2}} - \left(\frac{\arcsin(v/c)}{(v/c)} - 1\right) + \right.$$
$$\left.\left(\frac{1}{\sqrt{1-(v^2/c^2)\xi^2}} - 1\right)\right]\frac{q}{\mathrm{r}} - \frac{1}{1-v^2/c^2}\frac{v^2}{c^2}\left(\frac{1}{3} - \frac{2C_\delta v^2}{15c^2}\right)\frac{q}{\mathrm{r}} = 0 \;\Longrightarrow$$
$$\left(\frac{1}{\sqrt{1-(v^2/c^2)}} - \frac{\arcsin(v/c)}{(v/c)}\right)\frac{q}{\mathrm{r}} - \frac{1}{1-v^2/c^2}\frac{v^2}{c^2}\left(\frac{1}{3} - \frac{2C_\delta v^2}{15c^2}\right)\frac{q}{\mathrm{r}} = 0. \tag{8}$$

But the above equation cannot be fulfilled which is verified by expansion of Eq. (8) in series over $(v/c)^2$

$$\frac{v^4}{c^4}\frac{4C_\rho - 1}{30}\frac{q}{\mathrm{r}} + \mathcal{O}\left(\frac{v^6}{c^6}\right) = 0 \;, \tag{9}$$

Because calculations in *Appendix B* are made with accuracy to $(v/c)^6$ and other quantities entering into Eq. (8) are calculated exactly, violation of Eq. (9) in fourth order of (v/c) unambiguously indicates non-equivalence of the gauges in the case of uniformly moving charge

It should be noted that while calculating I_∂, we avoid to specify a form of the function describing the charge density. In contrast to our approach, Hnizdo should make some specific choice of the function ρ to obtain equivalence of the potentials created by the uniformly moving charge. In his work [3], the elementary charge is treated as a charged sphere made of the material of ideal conductivity. In motion, this sphere contracts in accordance to the Lorentz transformations.

Formally, the results of [2, 3] say in favor of identity of the expressions for the Coulomb– and Lorenz–gauge potentials. But to obtain it, Hnizdo uses operation of direct differentiation of Φ_{C}, written in the form (Eq. (7) of [3]),

$$\Phi_{\mathrm{C}}(\boldsymbol{r}) = \begin{cases} \arctan[\beta s/\sqrt{\tfrac{1}{4}(r_1+r_2)^2 - \beta^2 s^2}]/\beta s & \text{for } \gamma^2 x^2 + \zeta^2 \geq s^2 \\ \arctan(\beta\gamma)/\beta s & \text{for } \gamma^2 x^2 + \zeta^2 < s^2 \end{cases}, \tag{10}$$
$$r_{1,2} = \sqrt{x^2 + (\zeta \pm \beta s)^2} \qquad \zeta = \sqrt{y^2 + z^2} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}} \qquad \beta = \frac{v}{c}\;,$$

with respect to x, that is questionable.

Φ_C given by Eq. (10) is the surface potential represented by the integral (Ch.IV.5.6, Eq. (26) of [6])

$$\varphi(\boldsymbol{r}) = \iint \frac{(\boldsymbol{\sigma}(\boldsymbol{r}_P) \cdot \mathrm{d}\boldsymbol{s}_P)}{|\boldsymbol{r} - \boldsymbol{r}_P|} \ , \tag{11}$$

where $\boldsymbol{s}_P$ is the charged surface and $\boldsymbol{\sigma}(\boldsymbol{r}_P)$ the surface density of the charge. As it is shown in this chapter of [6], the surface potentials are represented by improper integrals at the points belonging to the surface $s = \sqrt{\gamma^2 x^2 + \zeta^2}$. The improper integral exists if the singularity in its integrand is integrable. The singularity $\frac{1}{|\boldsymbol{r}-\boldsymbol{r}_P|}$ is integrable over the surface of oblate spheroid where, in the model of Hnizdo, the charge is located, and, therefore, the surface potential exists too. The singularity of higher power, *i.e.* $\frac{\cos\phi}{|\boldsymbol{r}-\boldsymbol{r}_P|^2}$ appearing due to differentiation of the potential, is still integrable because the curvature of the spheroid is non-zero value on the whole surface S. But second differentiation gives the $\frac{\cos^2\phi}{|\boldsymbol{r}-\boldsymbol{r}_P|^3}$ singularity which is not integrable. So the second partial derivative of Φ_C does not exist in the model of Hnizdo. Therefore, his calculations presented in [3] are at least questionable.

Actually, the second derivative of Φ_C can be calculated in the model of the elementary charge as a conducting ellipsoid, *i.e.* when the charged region of the ellipsoid is its surface. But this quantity can be correctly calculated only if the surface charged layer is of finite thickness δl and after all evaluations, $\delta l \to 0$. But this approach changes calculations given in [3]. Moreover, the density of the elementary charge in *no observable* quantity within the classical electrodynamics. So while calculating Φ_φ one cannot specify a form of ρ. Calculations of [2, 3] are based on such a specification, which is wrong. In contrast to the above cited works, in our calculations (*Appendix B*) we do not use any specific form of ρ but we only give estimate in what range Φ_φ can vary depending on the form of ρ.

So finally we have that for the single charge uniformly moving along the x axis, the difference of the electric fields in the Coulomb and Lorenz gauges is

$$\boldsymbol{E}_\mathrm{C} - \boldsymbol{E}_\mathrm{L} \approx \frac{(4C_\rho - 1)}{30}\frac{v^4}{c^4}\boldsymbol{\nabla}\left[\frac{q}{\sqrt{\frac{(x-vt)^2}{1-v^2/c^2} + y^2 + z^2}}\right] \ ; \ C_\rho > 1 \ . \tag{12}$$

Now we consider another aspect of the gauges–equivalence theorem. After appearing work [1] it was published the paper of Jackson [8] where the procedure of transformation of the Coulomb–gauge to the Lorenz–gauge potentials is developed. This procedure is based on introducing the "gauge functions" $\chi_{\rm C}$ and Ψ with the properties

$$\frac{\partial \chi_{\rm C}}{c\partial t} = [\Phi_{\rm C} - \Phi_{\rm L}] \quad , \quad \nabla\Psi = [\boldsymbol{A}_{\rm L} - \boldsymbol{A}_{\rm C}]$$

If these functions are equal each to the other, equivalence of the fields in different gauges follows with necessity

$$\nabla \frac{\partial \chi_{\rm C}}{c\partial t} = \nabla[\Phi_{\rm C} - \Phi_{\rm L}] = \frac{\partial}{c\partial t}\nabla\Psi = \frac{1}{c}\frac{\partial}{\partial t}[\boldsymbol{A}_{\rm L} - \boldsymbol{A}_{\rm C}] \quad \Longrightarrow$$
$$\boldsymbol{E}_{\rm C} = -\nabla\Phi_{\rm C} - \frac{1}{c}\frac{\partial \boldsymbol{A}_{\rm C}}{\partial t} = -\nabla\Phi_{\rm L} - \frac{1}{c}\frac{\partial \boldsymbol{A}_{\rm L}}{\partial t} = \boldsymbol{E}_{\rm L}\ . \qquad (13)$$

But [8] contains one missing point in Sec. IV where the author derives the function Ψ and associates this function with the "gauge function" of Sec. III. A final form (4.4) of Ψ in [8] is equal to the form (3.6) of the other "gauge function" $\chi_{\rm C}$. However, a procedure of derivation of Eq. (4.4) staring from Eq. (4.1) of [8] is questionable. Let us consider this procedure in detail.

Actually, Eq. (4.1) of [8]

$$\Psi(\boldsymbol{r},t) = \frac{1}{4\pi c}\int {\rm d}^3r''\frac{1}{R'}\left[\int {\rm d}^3r'\frac{1}{R''}\frac{\partial}{\partial t'}\varrho(\boldsymbol{r}',t')\right]_{ret} , \qquad (14)$$

where $\boldsymbol{R}' = \boldsymbol{r} - \boldsymbol{r}''$, $\boldsymbol{R}'' = \boldsymbol{r}' - \boldsymbol{r}''$ and "ret" means $t' = t - R'/c$, is a solution of an inhomogeneous wave equation for Ψ (the first of Eqs. (2.10) of [8])

$$\nabla^2\Psi - \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} = -\frac{1}{c}\int {\rm d}^3r'\frac{1}{R}\frac{\partial}{\partial t}\varrho(\boldsymbol{r}',t)\ , \qquad (15)$$

The term in the *rhs* of Eq. (15) is easily calculated

$$\int {\rm d}^3r'\frac{1}{cR}\frac{\partial}{\partial t}\varrho(\boldsymbol{r}',t) = \frac{\partial}{c\partial t}\int {\rm d}^3r'\frac{\varrho(\boldsymbol{r}',t)}{R} = \frac{\partial}{c\partial t}\frac{Q}{|\boldsymbol{r} - \boldsymbol{r}_0(t)|}$$
$$= \frac{Q}{c}\frac{\boldsymbol{v}_0(t)\cdot(\boldsymbol{r} - \boldsymbol{r}_0(t))}{|\boldsymbol{r} - \boldsymbol{r}_0(t)|^3}\ , \qquad (16)$$

where $\boldsymbol{r}_0(t)$ and $\boldsymbol{v}_0(t)$ are the coordinate and velocity of the center of the charge Q. We note that the *rhs* of the above equation is the partial time derivative of the Coulomb-gauge scalar potential, which coincides to the *rhs* of Eq. (3.8) of [8]. So Eq. (14) can be presented as

$$\Psi(\boldsymbol{r},t)=\frac{Q}{4\pi c^2}\int \mathrm{d}^3r''\frac{1}{R'}\left[\frac{\boldsymbol{v}_0(t')\cdot(\boldsymbol{r}''-\boldsymbol{r}_0(t'))}{|\boldsymbol{r}''-\boldsymbol{r}_0(t)|^3}\right]_{t'=t-|\boldsymbol{r}-\boldsymbol{r}''|/c} . \quad (17)$$

Because the integral (17) is of retarded type, it cannot be calculated in the closed form for arbitrary motion of the charge. But if the charge moves uniformly, for example, along the x axis with the velocity v, this integral is calculated in accordance to the rules given in [9] (Ch. 5, Eq. (5-2.4))

$$\Psi(\boldsymbol{r},t)=\frac{Q}{4\pi c^2}\int\frac{\mathrm{d}x''\mathrm{d}y''\mathrm{d}z''}{\sqrt{(x-vt-x'')^2+b^2\left[(y-y'')^2+(z-z'')^2\right]}}\times \frac{vx''}{\left[x''^2+y''^2+z''^2\right]^{\frac{3}{2}}} , \quad (18)$$

with $b=\sqrt{1-v^2/c^2}$. It is an obvious point that the integral (44) diverges for $r''=\sqrt{x''^2+y''^2+z''^2}$, $r''\to\infty$ as

$$\Psi(\boldsymbol{r},t)\sim\lim_{L\to\infty}\int^{L}\frac{\mathrm{d}r''}{r''}=\lim_{L\to\infty}\ln L=\infty . \quad (19)$$

We note that this divergence is non-removable which means that the function Ψ cannot exist so the calculation procedure from Eq. (14) to Eq. (13) is incorrect too.

But if one calculates Ψ from Eq. (4.4) of [8], one finds that this quantity is finite. The error of Jackson is in the incorrect performing of the *retarded* integral (4.1).

The integration of the retarded quantities is not reduced by substituting $t_{ret}=t-\frac{|\boldsymbol{r}-\boldsymbol{r}''|}{c}$ instead of t' only. Still Liénard in his original work [10] on calculation of the potentials of the moving charge notes (cited from p. 98 of [4]) that "because of the motion of the charge, the region of space from which the charge "sends" electric and magnetic field signals is not the same as the volume occupied by the stationary charge". According to Liénard, if the region occupied by the stationary charge is Ω, then the region to be extended over the region $\Omega/\left[1-(\boldsymbol{v}\cdot\boldsymbol{n})/c\right]$, $\boldsymbol{n}=\boldsymbol{r}/r$, where $\boldsymbol{v}$ is the velocity of the charge and r

the distance from the charge to the point of observation. So even if the actual volume of the charge is "infinitesimal", the volume of integration is not. In fact, according to Eq. (3-1.8) of [9], it can be *infinitely large*, if the velocity of the charge is equal to the speed of light and if the charge moves toward the point of observation. In Eq. (14), the retarded variable is just $\boldsymbol{r}''$ and, therefore, each infinitesimal element of integration $\mathrm{d}\boldsymbol{r}''$ must be changed by

$$\mathrm{d}^3r'' \to \mathrm{d}^3r''_{ret} = \frac{\mathrm{d}^3r''}{[1 - \boldsymbol{v}\cdot(\boldsymbol{r}-\boldsymbol{r}'')/(c|\boldsymbol{r}-\boldsymbol{r}''|)]} \ . \tag{20}$$

where now $\boldsymbol{v}$ is the velocity of the charge at retarded instant of time. We emphasize that this operation must be applied to *each* element of integration which is caused by two points, *i.e.*

- even if a region occupied by the retarded source (in our case, the classical charge) is subtended to a single point, the location of this point is unknown so integration should be made over the whole space;
- range of definition of the mathematical function ρ, describing the density of the classical charge, is the whole space so each element of this range of definition must be transformed in accordance to (20).

The above consideration allows us to find the error in calculations of Sec. IV of [8]. Transformation of Eq. (4.1) to Eq. (4.2) of this paper

$$\Psi(\boldsymbol{r},t) = \frac{1}{4\pi c}\int \mathrm{d}^3r' \int \mathrm{d}^3R' \frac{1}{R'}\frac{1}{|\boldsymbol{R}'-\boldsymbol{R}|}\frac{\partial}{\partial t'}\varrho(\boldsymbol{r}', t - R'/c) \ , \tag{21}$$

is still correct. But this expression is written in symbolic form where the rules of integration are not defined. When we proceed to make integration over the $\boldsymbol{R}'$ coordinate, we should introduce the correcting factor, which appears due to the retardation type of the integral, into d^3R' because now the retarded coordinate is $\boldsymbol{R}'$,

$$\Psi(\boldsymbol{r},t) = \frac{1}{4\pi c}\int \mathrm{d}^3r' \int \frac{\mathrm{d}^3R'}{[1-(\boldsymbol{v}(t')\cdot\boldsymbol{R}')/(cR')]}\frac{1}{R'}\frac{1}{|\boldsymbol{R}'-\boldsymbol{R}|}\frac{\partial}{\partial t'}\varrho(\boldsymbol{x}', t - R'/c) \ . \tag{22}$$

So instead of Jackson's calculations from Eq. (4.2) (Eq. (21) of this work) to Eq. (4.3), *i.e.* integration over the angular part of d^3R'

$$\frac{1}{4\pi c}\int \mathrm{d}^3r' \int \mathrm{d}^3R' \frac{1}{R'}\frac{1}{|\boldsymbol{R'}-\boldsymbol{R}|}\frac{\partial}{\partial t'}\varrho(\boldsymbol{r'},t-R'/c) =$$
$$\frac{1}{4\pi c}\int \mathrm{d}^3r' \int \frac{R'^2\mathrm{d}R'}{R'}\frac{\partial}{\partial t'}\varrho(\boldsymbol{r'},t-R'/c)\int \mathrm{d}\Omega' \frac{1}{|\boldsymbol{R'}-\boldsymbol{R}|} =$$
$$\frac{1}{c}\int \mathrm{d}^3r' \int \frac{R'^2\mathrm{d}R'}{R'}\frac{\partial}{\partial t'}\varrho(\boldsymbol{r'},t-R'/c)\frac{1}{r_>}\ , \quad (23)$$

where $r_>$ is the larger of R' and R, we should have

$$\frac{1}{4\pi c}\int \mathrm{d}^3r' \int \mathrm{d}^3R'_{ret} \frac{1}{R'}\frac{1}{|\boldsymbol{R'}-\boldsymbol{R}|}\frac{\partial}{\partial t'}\varrho(\boldsymbol{r'},t-R'/c) =$$
$$\frac{1}{4\pi c}\int \mathrm{d}^3r' \int \frac{\mathrm{d}^3R'}{[1-(\boldsymbol{v}(t')\cdot\boldsymbol{R'})/(cR')]}\frac{1}{R'}\frac{1}{|\boldsymbol{R'}-\boldsymbol{R}|}\frac{\partial}{\partial t'}\varrho(\boldsymbol{r'},t-R'/c) =$$
$$\frac{1}{4\pi c}\int \mathrm{d}^3r' \int \frac{R'^2\mathrm{d}R'}{R'}\frac{\partial}{\partial t'}\varrho(\boldsymbol{r'},t-R'/c)\int \frac{\mathrm{d}\Omega'}{[1-(\boldsymbol{v}(t')\cdot\boldsymbol{R'})/(cR')]}\frac{1}{|\boldsymbol{R'}-\boldsymbol{R}|} \neq$$
$$\frac{1}{c}\int \mathrm{d}^3r' \int \frac{R'^2\mathrm{d}R'}{R'}\frac{\partial}{\partial t'}\varrho(\boldsymbol{r'},t-R'/c)\frac{1}{r_>}\ , \quad (24)$$

because

$$I_\Omega = \frac{1}{4\pi}\int \frac{1}{|\boldsymbol{R'}-\boldsymbol{R}|}\frac{\mathrm{d}\Omega'}{\left[1-(\boldsymbol{v}(t')\cdot\boldsymbol{R'}/(cR')\right]} \neq \frac{1}{r_>}\ , \quad (25)$$

Clearly, Eq. (25) violates Jackson's derivation of the "gauge function" since integration over the angular variables (25) does not give a finite Eq. (23) for Ψ but an expression containing logarithmic divergence at infinity.

Thus, we conclude that both statements, *i.e.* (a) "the expressions for the Coulomb– and Lorenz–gauge electric fields are identical" and (b) "there exists a gauge function transforming the Coulomb–gauge potentials into the Lorenz–gauge potentials, and otherwise", are incorrect.

Appendix A. Derivation of I_φ.

The integral I_φ is of the form

$$I_\varphi(\boldsymbol{r}) = -\frac{qv^2}{4\pi c^2}\lim_{V_{ch}\to 0}\int\limits_{V_{ch}}^{\infty}\frac{3\kappa'^2 - r'^2}{\left[\sqrt{\kappa'^2+y'^2+z'^2}\right]^5}\times \frac{\mathrm{d}\kappa'\mathrm{d}y'\mathrm{d}z'}{\sqrt{(\kappa-\kappa')^2+b^2[(y-y')^2+(z-z')^2]}}\,, \quad (26)$$

To calculate it in the general case, first, we convert to the "contracted" coordinates (7) so (26) takes the form

$$I_\varphi(\boldsymbol{r}) = -\frac{qv^2}{4\pi c^2}\lim_{V_{ch}\to 0}\int\limits_{V_{ch}}^{\infty}\frac{(2b^2+1)\mathrm{x}'^2 - \mathrm{r}'^2}{\left(\sqrt{b^2x'^2+y'^2+z'^2}\right)^5}\times \frac{b\mathrm{d}\mathrm{x}'\mathrm{d}\mathrm{y}'\mathrm{d}\mathrm{z}'}{b\sqrt{(\mathrm{x}-\mathrm{x}')^2+(\mathrm{y}-\mathrm{y}')^2+(\mathrm{z}-\mathrm{z}')^2]}}\,, \quad (27)$$

Because the charge acquires spherical shape in new coordinates as the charge being at rest, the volume occupied by the charge is a sphere $V_{ch} = (4\pi r_{ch}^3/3)$, where r_{ch} is the radius of the elementary charge. Integral (27) re-written in the spherical coordinates

$$\mathrm{x} = \mathrm{r}\cos\theta \quad \mathrm{y} = \mathrm{r}\sin\theta\cos\phi \quad \mathrm{z} = \mathrm{r}\sin\theta\sin\phi$$

is

$$I_\varphi(\boldsymbol{r}) = -\frac{qv^2}{4\pi c^2}\lim_{r_{ch}\to 0}\int\limits_{r_{ch}}^{\infty}\frac{\mathrm{dr}'}{\mathrm{r}'}\int\limits_0^{\pi}\sin\theta'\mathrm{d}\theta'\int\limits_0^{2\pi}\frac{3\cos^2\theta' - 1 - \frac{2v^2}{c^2}\cos^2\theta'}{[1-(v^2/c^2)\cos^2\theta']^{5/2}}\frac{\mathrm{d}\phi'}{|\,\mathbf{r}-\mathbf{r}'|}. \quad (28)$$

One can expand the term $\frac{1}{|\,\mathbf{r}-\mathbf{r}'|}$ in series over the spherical harmonics using Eqs. (3.41) and (3.68) of [7]

$$\frac{1}{|\,\mathbf{r}-\mathbf{r}'|} = \sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\gamma) = \sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}\Big\{P_l(\cos\theta)P_l(\cos\theta') + 2\sum_{m=1}^{l}\frac{(l-m)!}{(l+m)!}P_l^m(\cos\theta)P_l^m(\cos\theta')\cos[m(\phi-\phi')]\Big\}\,, \quad (29)$$

where $r_<$ ($r_>$) is the smaller (larger) of $|\mathbf{r}|$ and $|\mathbf{r}'|$, and γ is the angle between $\mathbf{r}$ and $\mathbf{r}'$. After integration over ϕ' the double sums containing associated Legendre functions $P_l^m(\cos\theta')$ are eliminated and only the terms containing $P_l(\cos\theta')$ survive. Therefore, Eq. (28) reduces to

$$I_\varphi(\boldsymbol{r}) = -\frac{qv^2}{2c^2}\lim_{r_{ch}\to 0}\sum_{l=0}^{\infty}\int_{r_{ch}}^{\infty}\frac{\mathrm{d}\mathrm{r}'}{\mathrm{r}'}\int_{-1}^{+1}\mathrm{d}\xi'\frac{r_<^l}{r_>^{l+1}}P_l(\xi)P_l(\xi')\frac{3\xi'^2-1-2(v^2/c^2)\xi'^2}{\left[1-(v^2/c^2)\xi'^2\right]^{5/2}} . \qquad (30)$$

with $\xi = \cos\theta$. Because the radial and angular variables are separated in the above integral, the latter can be calculated in the closed form. To do it, one needs to expand the integrand of (30) into a series over $(v/c)^2$ and, using a property of orthogonality of the Legendre polynomials, to make integration on ξ' and then on r$'$ variables. It is possible to verify by means of *Mathematica* software, for example, that the obtained result is equal to expansion of the expression

$$I'_\varphi(\boldsymbol{r}) = -\frac{q}{\mathrm{r}}\left[\left(\frac{\arcsin(v/c)}{(v/c)}-1\right)-\left(\frac{1}{\sqrt{1-(v^2/c^2)\xi^2}}-1\right)\right] , \qquad (31)$$

in a series over $(v/c)^2$. So we conclude that $I_\varphi(\boldsymbol{r})$ is presented by Eq. (31).

Appendix B. Derivation of I_∂

To find $\partial^2\Phi_\mathrm{C}/\partial x^2$ at interior points of the charge, first we derive expression for the potential which do not lead to divergences in this region.

Before, we determine functional dependence of the charge distribution on the radial variable. It follows from the Lorentz contraction of the moving bodies that the charge, which shape is spherical in co-moving frame, should acquires ellipsoidal shape in the laboratory frame. Because in the co-moving frame the layers of equal potential inside the charge are concentric spheres, the Lorentz transformations convert the spherical layers into the layers of ellipsoidal shape and this conversation can be written in the functional form as

$$\rho_{co}(x^2+y^2+z^2) \to \rho_{lab}\left(\frac{(x-vt)^2}{1-v^2/c^2}+y^2+z^2\right) .$$

The potential in the interior region of the charge is, therefore,

$$\Phi_\mathrm{C} = \int_{V_{ch}}\frac{\rho\left(\frac{(x-vt)^2}{1-v^2/c^2}+y^2+z^2\right)}{|\boldsymbol{R}-\boldsymbol{r}|}\mathrm{d}\boldsymbol{r} . \qquad (32)$$

After transformations (7), Eq. (32) takes a form

$$\Phi_{\rm C} = \int_{V_{ch}} \frac{\rho\,({\rm r})}{\sqrt{(1 - v^2/c^2)({\rm X} - {\rm x})^2 + ({\rm Y} - {\rm y})^2 + ({\rm Z} - {\rm z})^2}} {\rm d}^3{\rm r}\ . \tag{33}$$

and the volume of integration of elliptical shape transforms to the volume of spherical shape. We shall study a limit $v \ll c$ so we expand $\Phi_{\rm C} \approx \Phi_0 + (v/c)^2\Phi_2 + (v/c)^4\Phi_4 + \ldots$ It is easily seen from the Gauss theorem and symmetry of $\rho({\rm r})$ that zeroth term

$$\Phi_0 = \int_{V_{ch}} \frac{\rho\,({\rm r})}{\sqrt{({\rm X} - {\rm x})^2 + ({\rm Y} - {\rm y})^2 + ({\rm Z} - {\rm z})^2}} {\rm d}^3{\rm r}\ . \tag{34}$$

depends on ${\rm R} = \sqrt{{\rm X}^2 + {\rm Y}^2 + {\rm Z}^2}$ only. In this approach, the second derivative of Φ_0 contains no divergence. The second term is

$$\Phi_2 = \frac{v^2}{2c^2} \int_{V_{ch}} \frac{({\rm X} - {\rm x})^2 \rho\,({\rm r})}{\left[\sqrt{({\rm X} - {\rm x})^2 + ({\rm Y} - {\rm y})^2 + ({\rm Z} - {\rm z})^2}\right]^3} {\rm d}^3{\rm r}\ . \tag{35}$$

Expanding the above integrand into the spherical harmonics and using the spherical symmetry of ρ in the contracted coordinates, one should obtain that $\Phi_2 = (v^2/2c^2)\cos^2\theta({\rm r}\partial\Phi_0({\rm r})/\partial{\rm r})$ (long but trivial calculations are omitted here) so we have for $\Phi_{\rm C}$ calculated at interior points of the charge with accuracy to $(v/c)^4$ terms

$$\Phi_{\rm C}(\mathbf{r}) = \Phi_0({\rm r}) - \frac{v^2}{2c^2}\cos^2\theta\left({\rm r}\frac{\partial\Phi_0({\rm r})}{\partial{\rm r}}\right)\ . \tag{36}$$

Eq. (36) does not yield us a solution of the problem yet because actually it is impossible to find $\Phi_0({\rm r})$ while ρ in Eq. (35) is given in the general form. But we know that, for any realization of ρ, the volume integral of this function is equal to q so if we find a relation connecting derivative of $\Phi_0({\rm r})$ with ρ, the problem can be solved.

With regard to symmetry of the problem, the total derivative of $\Phi_0({\rm r})$ should contain both angular and radial derivatives. But if the radius of the elementary charge tends to zero, the radial derivatives are becoming to be the majorants with respect to the angular ones so we shall seek $\partial^2\Phi_{\rm C}/\partial x^2$ as a function of the radial derivatives only.

The second derivative $\partial^2\Phi_{\rm C}/\partial x^2$, re-written via the r variable, is

$$\frac{\partial^2\Phi_{\rm C}}{\partial x^2} = \left(\frac{\partial{\rm r}}{\partial x}\right)^2 \frac{\partial^2\Phi_{\rm C}}{\partial{\rm r}^2} + \frac{\partial^2{\rm r}}{\partial x^2}\frac{\partial\Phi_{\rm C}}{\partial{\rm r}}\ , \tag{37}$$

Using the Poisson equation in the spherical coordinates (we assume that the dominant dependence is that one on r and the angular terms of the Laplacian are omitted)

$$\left[\frac{\partial^2\Phi_{\rm C}}{\partial {\rm r}^2}+\frac{2}{\rm r}\frac{\partial\Phi_{\rm C}}{\partial {\rm r}}\right]=-4\pi\rho(\mathbf{r})\ , \tag{38}$$

we have from both (37) and (38)

$$\frac{\partial^2\Phi_{\rm C}}{\partial x^2}=\left(\frac{\partial {\rm r}}{\partial x}\right)^2\frac{\partial^2\Phi_{\rm C}}{\partial {\rm r}^2}+\frac{2}{\rm r}\left(\frac{\partial {\rm r}}{\partial x}\right)^2\frac{\partial\Phi_{\rm C}}{\partial {\rm r}}+\frac{\partial^2 {\rm r}}{\partial x^2}\frac{\partial\Phi_{\rm C}}{\partial {\rm r}}-\frac{2}{\rm r}\left(\frac{\partial {\rm r}}{\partial x}\right)^2\frac{\partial\Phi_{\rm C}}{\partial {\rm r}}\quad\rightarrow$$

$$\frac{\partial^2\Phi_{\rm C}}{\partial x^2}=-4\pi\left(\frac{\partial {\rm r}}{\partial x}\right)^2\rho(\mathbf{r})+\frac{1}{1-v^2/c^2}\left[\frac{{\rm r}^2-3\left(\frac{x^2}{1-v^2/c^2}\right)}{{\rm r}^3}\right]\frac{\partial\Phi_{\rm C}}{\partial {\rm r}}\ . \tag{39}$$

Using Eq. (36), we have for Eq. (39)

$$\begin{aligned}\frac{\partial^2\Phi_{\rm C}}{\partial x^2}&=-4\pi\frac{\cos^2\theta}{1-v^2/c^2}\rho(\mathbf{r})+\frac{1-3\cos^2\theta}{\rm r}\frac{\partial\Phi_0}{\partial {\rm r}}\\&\quad+\frac{v^2\left[1-3\cos^2\theta\right]\cos^2\theta}{2(1-v^2/c^2)c^2}\left(\frac{1}{\rm r}\frac{\partial}{\partial {\rm r}}\left({\rm r}\frac{\partial\Phi_0({\rm r})}{\partial {\rm r}}\right)\right)\ .\end{aligned} \tag{40}$$

where $\cos\theta=\frac{x/\sqrt{1-v^2/c^2}}{\rm r}$. Because the *lhs* of the above equation should be further integrated in the region of spherical shape, the second term in its *rhs* will be equal to zero so we omit it from the following consideration.

Now we need to estimate result of integration of the term $\frac{1}{\rm r}\frac{\partial}{\partial {\rm r}}\left({\rm r}\frac{\partial\Phi_0({\rm r})}{\partial {\rm r}}\right)$ in the interior region of the charge. By means of zeroth approximation of Eq. (38), we have for the term in question

$$\begin{aligned}&\frac{1}{\rm r}\frac{\partial}{\partial {\rm r}}\left({\rm r}\frac{\partial\Phi_0({\rm r})}{\partial {\rm r}}\right)=\frac{\partial^2\Phi_0({\rm r})}{\partial {\rm r}^2}+\frac{1}{\rm r}\frac{\partial\Phi_0({\rm r})}{\partial {\rm r}}=-\frac{1}{\rm r}\frac{\partial\Phi_0({\rm r})}{\partial {\rm r}}-4\pi\rho(\mathbf{r})\quad\Longrightarrow\\&\left|\frac{1}{\rm r}\frac{\partial}{\partial {\rm r}}\left({\rm r}\frac{\partial\Phi_0({\rm r})}{\partial {\rm r}}\right)\right|>4\pi\rho=\left|\frac{1}{{\rm r}^2}\frac{\partial}{\partial {\rm r}}\left({\rm r}^2\frac{\partial\Phi_0({\rm r})}{\partial {\rm r}}\right)\right|\ ;\ 0<{\rm r}<{\rm r}_{\rm ch}\ .\end{aligned} \tag{41}$$

Further calculation requires a definition of the form of the function ρ, which contradicts the concept of the elementary charge. Because within the classical electrodynamics, the internal structure of the elementary charge is not observable in principle, such a definition of the form of ρ is impossible. But because the term $\frac{1}{\rm r}\frac{\partial}{\partial {\rm r}}\left({\rm r}\frac{\partial\Phi_0({\rm r})}{\partial {\rm r}}\right)$ should be integrated interior the charge, where the term $4\pi\rho$ is integrated too, we are able to assume for $\mathrm{r_{ch}} \to 0$

$$\frac{1}{\mathrm{r}}\frac{\partial}{\partial \mathrm{r}}\left(\mathrm{r}\frac{\partial \Phi_0(\mathrm{r})}{\partial \mathrm{r}}\right) = -4C_\rho \pi \rho(\mathbf{r}) \quad ; \quad C_\rho > 1 \ , \tag{42}$$

having in mind that this relation will be used only in the integrand calculating over $\mathrm{r} \in [0, \mathrm{r_{ch}}]$. Thus, we have overcome the main difficulty in the given procedure, *i.e.* we have obtained some estimate for the second partial derivative of the scalar potential interior the charge without and specifying the form of the function describing this charge. Now, the second derivative of Φ_C interior the change reads as

$$\frac{\partial^2 \Phi_\mathrm{C}}{\partial x^2} = -4\pi \frac{\cos^2\theta}{1 - v^2/c^2}\left[1 + \frac{v^2(1 - 3\cos^2\theta)C_\rho}{2c^2}\right]\rho(\mathbf{r}) \ , \tag{43}$$

where the density of the elementary charge, which is of spherical symmetry ($\rho(\mathbf{r}) = \rho(\mathrm{r})$) in the contracted coordinates, is connected with the value of the charge as

$$\int \rho(\mathbf{r}) \mathrm{d}\mathbf{r} = \int \rho(\mathrm{r})\mathrm{r}^2 \mathrm{d}\mathrm{r}\mathrm{d}\Omega = 4\pi \int \rho(\mathrm{r})\mathrm{r}^2 \mathrm{d}\mathrm{r} = q \ . \tag{44}$$

By means of (43), the quantity I_∂ is easily evaluated

$$I_\partial = \frac{v^2}{4\pi c^2} \lim_{V_{ch}\to 0} \int_0^{V_{ch}} \left[\frac{\partial^2 \Phi_\mathrm{C}}{\partial \kappa'^2}\right]_{in} \frac{\mathrm{d}\kappa' \mathrm{d}y' \mathrm{d}z'}{\sqrt{(\kappa - \kappa')^2 + b^2\left((y - y')^2 + (z - z')^2\right)}} =$$

$$\frac{v^2}{c^2 \mathrm{r}} \lim_{\mathrm{r}_{ch}\to 0} \int_0^{\mathrm{r}_{ch}} \int \mathrm{d}\Omega' \frac{\cos^2\theta'}{1 - v^2/c^2}\left[1 + \frac{C_\rho v^2}{2c^2}\left(1 - 3\cos^2\theta'\right)\right]\rho(\mathrm{r}')\mathrm{r}'^2 \mathrm{d}\mathrm{r}' =$$

$$\frac{q}{1 - v^2/c^2}\frac{v^2}{c^2 \mathrm{r}}\left(\frac{1}{3} - \frac{2C_\rho v^2}{15c^2}\right) . \tag{45}$$

References

[1] V. V. Onoochin, “On non-equivalence of Lorentz and Coulomb gauges within classical electrodynamics”, *Ann. Fond. L. Broglie*, **27**, 163–183, 2002
(*arXiv.org e-print* physics/0111017)

[2] V. Hnizdo, “Potentials of a uniformly moving point charge in the Coulomb gauge”, *Eur. J. Phys.*, **25**, 351–360, 2005
(*arXiv.org e-print* physics/0307124)

[3] V. Hnizdo, "Regularization of the second-order partial derivatives of the Coulomb potential of a point charge" (*arXiv.org e-print* physics/0409072)

[4] O. D. Jefimenko, *Electricity and Magnetism*, (Electret Scientific Co., Star City, 1997).

[5] W. K. H. Panofsky and M. Phillips, *Classical Electricity and Magnetism*, (Addison Wesley, London, 1964).

[6] A. N. Tikhonov and A. A. Samarski, *Equations of Mathematical Physics*, (Dover Publications, Inc., New York, 1990).

[7] J. D. Jackson, *Classical Electrodynamics* 2^{nd} edn (New York: Wiley, 1975).

[8] J. D. Jackson, "From Lorenz to Coulomb and other explicit gauge transformations", *Am. J. Phys.* **70**, 917 (2002).

[9] O.D. Jefimenko, *Electromagnetic Retardation and Theory of Relativity*, 2^{nd} edn., (Electret Scientific Co., Star City, 2004).

[10] A. Liénard, L'Éclairage Électrique, **16**, (1898) pp. 5, 53, 106 .

(Manuscrit reçu le 1er novembre 2005)